Modeling Uncertainties in Soil Properties by Random Finite Element Method

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Abstract

The Finite Element Method offers great potential for implementing uncertainties in properties of materials such as soil. In many cases, soil parameters should be considered as random variables or random fields. This paper discusses the Random Finite Element Method in predicting the variation of the elastic settlement due to the uncertainties of the elastic modulus. The geometry and the Poisson’s ratio are considered as invariable parameters. The correlation length for the 30×60 m field is considered 30 m in order to have a conservative estimation of the elastic modulus variation. The number of 400000 realizations of the random field are generated. By generating and analyzing multiple realizations, the statistics and the density function of the maximum settlement is estimated. The reliability of the foundation according to limit state failure, in the form of excessive settlements is estimated.

Keywords: Finite Element Method, Random Finite Element Method, Random Field, Monte Carlo Simulation, Elastic Settlement.

1. INTRODUCTION

Once a footing has been designed and constructed, the actual settlement it experiences on a real soil mass can be quite different than expected, due to the soil's spatial variability. The deterministic Finite Element Method (FEM) has been increasingly used in practise, especially for geotechnical problems with complex geometries and parameters. Unfortunately, the validity of the results obtained using this technique can be drastically limited by the variability of the parameters introduced in the model. Variability often leads to uncertainty. We do not know what a typical parameter of a soil is at a particular location unless we have measured it at that location. As we know a parameter varies from point to point in a soil layer and it has a spatial correlation.

The basic representation of uncertain parameters in the FEM model is obtained by introducing random variables or fields. Coupling probability theory with the FEM cannot eliminate uncertainties but can indeed provide a quantitative evaluation of their influence on the reliability of the results of the analysis. Random Finite Element Method (RFEM) has been applied to various kinds of geotechnical problems including two-dimensional slope stability [1-3], differential settlement due to tunnelling [4], infiltration analysis with a spatially varying permeability function [5] and so many other problems. The RFEM attempts to understand the effect of property variations and the spatial correlation between the parameters. RFEM uses many individual realizations of uncertain properties within an FEM, combined with the Monte Carlo method, to determine the probability of an undesired event. The method has found widespread use in the field of geotechnics, as it is relatively simple to be implemented and provides a comprehensive analysis of the effects of variability.

In this paper we take a problem of elastic foundation settlement and critically examine the applicability of RFEM in the context of reliability analysis. The paper considers the case of a single footing and estimates the Probability Density Function (PDF) of the maximum elastic settlement by the Monte Carlo simulation (MCs). This elastic soil layer is underlain by bedrock. A superstructure founded on this soil mass is idealized as a uniform pressure applied over the free surface. Only the soil elasticity is considered to be
spatially random. Uncertainties arising from the loads, geometry and the Poisson’s ratio are not considered. In addition, the soil is assumed to be isotropic—that is, the correlation structure is assumed to be the same in both the horizontal and vertical directions. A plane strain analysis is carried out. The deterministic FEM code and the maximum elastic settlement calculated by this code is compared with a FEM code, similar to that used in references [6]. For a footing of finite dimension, the 2-D model is admittedly just an approximation. However, the approximation is considered reasonable since the elastic modulus field is averaged in the z direction.

2. RANDOM FIELD SIMULATION

A random field $H(x, \theta)$ is a collection of random variables associated with a continuous index $x \in \Omega \subset R^n$, where $\theta \in \Theta$ is the coordinate in the outcome space. With this notation, $H(x_0, \theta)$ denotes a particular realization of the field, whereas $H(x_0, \theta)$ is the random variable associated with point $x_0$. Gaussian random fields are of practical interest because they are completely described by a mean function $\mu(x)$, a variance function $\sigma^2(x)$ and an autocorrelation coefficient function $\rho(x,x')$. Discretizing the random field $H(x_0, \theta)$ consists in approximating it by $\hat{H}(x_0, \theta)$, which is defined by means of a finite set of random variables $\{X_i, i=1,\ldots,n\}$, grouped in a random vector denoted by $X$ [6].

$$H(x_0, \theta) \longrightarrow \hat{H}(x_0, \theta)$$ (1)

Assume that the mean and covariance structure for the random field to be modeled has been established. Since the mean and covariance completely specifies a jointly normal distribution, in the discussion of this paper, we will only generate normally distributed random fields. Non normal random fields can often be obtained through a suitable transformation of a normally distributed field. We assume two simplifications in order to make our analysis easy, these two simplifications are stationarity and isotropy of the random field [7].

2.1 CORRELATION LENGTH

A convenient measure of the variability of a random field is the correlation length $\theta$, also sometimes referred to as the scale of fluctuation. Loosely speaking $\theta$ is the distance within which points are significantly correlated. Mathematically, $\theta$ is defined as the area under the correlation function. Conversely, two points separated by a distance more than $\theta$ will be largely uncorrelated [7].

2.2 MARKOV CORRELATION FUNCTION

The Markov correlation function is very commonly used because of its simplicity. Its correlation function has the form:

$$\rho(\tau) = \exp \left\{ -\frac{2|\tau|}{\theta} \right\}$$ (2)

where $\theta$ is the correlation length and $\tau$ is the distance between two typical points. The parameter $\theta$ can be interpreted as the separation distance beyond which the random field is largely uncorrelated.

2.3 COVARIANCE MATRIX DECOMPOSITION

Many methods have been presented for the generation of random fields of random values in one or more dimensions. In this paper we use the Covariance Matrix Decomposition method for simulating our random field. A brief explanation of this method is as follows [8].
The simulation of a normally distributed random field starts with the generation of a sequence of independent standard normally distributed random variables (zero mean, unit variance). In MATLAB there is a function, \( \text{r}=\text{randn}(n) \), that returns an \( n \)-by-\( n \) matrix containing pseudorandom values drawn from the standard normal distribution. In the code written for this paper we use this function in order to generate standard normal random variables.

Once a sequence of independent standard normally distributed random variables are available, there are quite a number of different algorithms all designed to produce a random field as mentioned above. The random field to be simulated is represented by \( n \) points, \( x_1, x_2, \ldots, x_n \) and realizations of \( X_1, X_2, \ldots, X_n \) are desired at each point, with the correct mean and covariance structure on average. If \( \rho \) is the correlation matrix associated with these points, having components:

\[
\rho_{ij} = \frac{\text{Cov}[X_i, X_j]}{\sigma_X(x_i)\sigma_X(x_j)}
\]

\( \sigma_X \) is standard deviation of the random field. An \( n \times n \) matrix \( \rho_{ij} \) can be decomposed into the product of a lower triangular matrix and its transpose,

\[
LL^T = \rho_{ij}
\]

If \( \text{Cov} \{X_i, X_j\} \) is a positive definite covariance matrix with elements \( C_{ij} = C(\tau_{ij}) \), Given the matrix \( L \), a properly correlated (on average) standard normal random field can be obtained by linearly combining the independent standard normal variates in order to produce a mean zero discrete process \( G_i = G(x_i) \):

\[
G_i = \sum_{j=1}^{n} L_{ij}z_j \quad i = 1,2,\ldots,n
\]

\( Z \) is the vector of random variables created in the first step. Finally, the known mean and standard deviation can be reintroduced to yield realizations for \( X \), which, on average, will show the correct target statistics:

\[
X_i = \mu_X(x_i) + \sigma_{XA}(x_i)G_i
\]

Where \( \sigma_{XA} \) is the standard deviation averaged over the area of the element [7]. Once a realization of the random field \( X \) has been generated, it can be used as input to a deterministic analysis such as Finite Element. For example, \( X \) could be the Young Modulus field \( E \) which could be used to compute elastic settlements under a structure lying on a layer of soil.

3. RANDOM FINITE ELEMENT METHOD

The RFEM involves generating a random field of soil properties with controlled mean, standard deviation and spatial correlation length, which is then mapped onto a finite element mesh. A conventional elastic finite element analysis using these properties is then performed to compute the footing settlement or any other response of the system, after which the process is repeated many times using MCS. In the MCS process, the underlying statistics of the properties are held constant, however the computed settlement of the footing under a constant load is different at each simulation [9]. In the current study 400,000 realizations were used for the parametric combination, leading in each case to an estimate of the mean and standard deviation of the footing settlement. The general Monte Carlo approach has the advantage of generating the entire distribution of output events or of estimating probabilities directly by counting the number of times the event occurs as a proportion of the total number of simulations [10].

A specific settlement design problem will be considered here in order to investigate the settlement probability distribution of footings designed against excessive settlement. The problem considered is on the surface of a two-dimensional linearly elastic soil mass underlain by bedrock.
4. **RESULTS**

We consider an elastic soil layer of thickness $t$ lying on a rigid substratum. A superstructure to be founded on this soil mass is idealized as a uniform pressure $P$ applied over a length $2B$ of the free surface as shown in Figure 1.

![Figure 1. Settlement of the foundation-problem definition](image)

The soil is modeled a linearly elastic isotropic material. A plane strain analysis is carried out. Due to the symmetry, we model half of the structure by finite elements. The parameters selected for the deterministic model are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soil layer thickness</td>
<td>$t$</td>
<td>30 m</td>
</tr>
<tr>
<td>Foundation width</td>
<td>$2B$</td>
<td>10 m</td>
</tr>
<tr>
<td>Applied pressure</td>
<td>$P$</td>
<td>0.2 MPa</td>
</tr>
<tr>
<td>Soil Young’s modulus</td>
<td>$E$</td>
<td>50 MPa</td>
</tr>
<tr>
<td>Soil Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Mesh width</td>
<td>$L$</td>
<td>60 m</td>
</tr>
</tbody>
</table>

The first step in our analysis was to find the deterministic elastic settlement given the soil elastic modulus everywhere equal to $\mu_E = 50$ MPa. This gives a maximum deterministic settlement of $\delta_{det} = 5.26$ cm. This result was compared to an example in K. Phoon, (2008) [6] in order to verify our results. The mesh and the maximum deterministic elastic settlement our shown in Figure 2. Our mesh contains 190 rectangular elements which are refined under the uniform pressure. The maximum settlement occurs under the foundation (point $A$) as shown in Figure 2 (b).

![Figure 2. a) Finite element mesh and b) the maximum deterministic elastic settlement, the dimensions are in meters](image)
The soil has two properties of interest to the settlement problem: the (effective) young modulus $E(x)$ and Poisson’s ratio $\nu(x)$, where $x$ is spatial position. Only the elastic modulus is considered to be a spatially random soil property. Poisson’s ratio was believed to have a smaller relative spatial variability and only a second-order importance to settlement statistics. It is held fixed at 0.3 over the entire soil mass for all realizations. Strictly speaking, there is no symmetry in the system when random fields of material properties are introduced. However, it is believed that this simplification does not significantly influence the results so we again consider half of the field.

The field of elastic modulus is assumed to follow a normal distribution so that $(E)$ is a Gaussian (normal) random field with mean $\mu_E$ and variance $\sigma_E^2$. A Markovian spatial correlation function, which gives the correlation coefficient between elastic modulus values at points separated by distance $\tau$, is used. The mean elastic modulus $\mu_E$ is held fixed at $5 \times 10^7$. The standard deviation of the elastic modulus is considered $1 \times 10^7$ and so its coefficient of variation equals to 0.2.

Realizations of the field of elastic modulus are produced using the Covariance Matrix Decomposition which was explained in the previous sections. The elastic modulus value assigned to the $i$th element is:

$$E_i = \mu_E(x_i) + \sigma_{EA}(x_i) G_i$$  \hspace{1cm} (7)

In which $\sigma_{EA}$ is the standard deviation of the elastic modulus multiplied by the variance reduction factor. Figure 3 shows a gray-scale representation of a possible realization of the elastic modulus field along with the finite element mesh.

Lighter areas denote larger values of $E(x)$ and darker areas denote the smaller values. This is just one possible realization of the elastic modulus field; the next realization could just as easily show the opposite trend. The Monte Carlo simulation adopted here involves the simulation of a realization of the elastic modulus field and subsequent finite-element analysis of that realization to yield a realization of the footing settlement(s). Repeating the process over a number of times generates a set of possible settlements from which distribution parameters can be estimated. A typical realization and the deformed mesh is shown in Figure 4.
In this study, 400000 realizations are performed for the input parameters \((\sigma_E, \mu_E, \text{ and } \theta_E)\). It can be assumed that the settlement for these realizations is approximately normally distributed, and \(\mu_s\) and \(\sigma_s^2\) are the mean and variance of the settlement. The mean and standard deviation for the number of 400000 realizations are equal to 5.57 cm and 0.76 cm respectively. We can plot the PDF for the settlement using Eq. 8.

\[
f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp \left\{ -\frac{1}{2} \left( \frac{x-\mu_x}{\sigma_x} \right)^2 \right\} \quad 0 \leq x < \infty
\]

The PDF plot of the maximum elastic settlement is plotted in Figure 5.

![Figure 5. The Probability Density Function of the maximum elastic settlement](image)

5. **RELIABILITY ANALYSIS BY RFEM**

The goal is to propose and investigate a reliability-based design methodology for the serviceability limit state of a footing settlement. The limit state function is defined in terms of the maximum settlement \(u_A\) at the center of the foundation:

\[
g(\xi) = u_0 - u_A(\xi)
\]

Where \(u_0\) is an admissible threshold initially set equal to 5 cm and \(\xi\) is the vector used for the random field discretization. Whenever the limit state function gives a negative value it shows that the unwanted settlement has occurred and is in the failure region and whenever is positive, we are in the safe region. In Figure 6 the negative and positive limit state functions resulting from each realization is plotted.

![Figure 6. 400000-realization MCs of \(E_p\). Points appearing in the safe and failure region corresponding to system failure.](image)
5.1 Probability of Failure

The estimated probability of failure is equal to the number of realizations which failed divided by the total number of realizations. So for a number of 400,000 realizations we have:

\[
I_i = \begin{cases} 
1 & \text{if } g(\xi) < 0 \\
0 & \text{otherwise}
\end{cases}
\]

For \( i = 1 \) to \( 400000 \), our estimate of \( P_f \) is simply:

\[
P_f = \frac{1}{n} \sum_{i=1}^{n} I_i
\]

\[
P_f = \frac{285723}{400000} = 0.7143
\]

This means that the probability that the maximum elastic settlement exceeds 5 cm is 71 percent.

7. Conclusion

A number of attempts have been made for modeling uncertainties in geotechnical parameters. In this paper, a Random Finite Element Method was used to estimate the reliability of the foundation according to limit state failure, in the form of excessive settlements. The selected random variable was the elastic modulus; the Poisson’s ratio, geometry and the pressure applied over the free surface were held constant. The settlement under the footing founded on this spatially random elastic modulus field of finite depth overlying bedrock, is presented by a normal distribution with parameters \( \mu \) and \( \sigma \). If \( E \) is also normally distributed. The results indicated that if an accurate description of the spatial variability is required, random fields may be employed. Their use in engineering problems requires discretization. In this paper a simple method has been presented for this purpose but more accurate methods are also available.

The Random Finite Element Method is ideal for estimating probabilities associated with the response of complex systems to random inputs, regardless of the amount of complexity. For such systems, simulation techniques can be well preformed since they are simple and lead to direct results.

8. References


