In the name of Allah, the Most Beneficent, the Most Merciful
Research Project

Title:

On the Asymptotic and Mittag-Leffler-Hyers-Ulam stability of fractional equations

By:

Nasrin Eghbali

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Nasrin Eghbali

Abstract. In this project, we define and investigate Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassiass stability of deterministic semilinear fractional Volterra integral equation. Also, we prove that this equation is stable with respect to the Chebyshev and Bielecki norms. The stability of stochastic systems driven by Brownian motion has also been studied.

Keywords: Mittag-Leffler-Hyers-Ulam stability; Mittag-Leffler-Hyers-Ulam-Rassiass stability; deterministic Volterra integral equation; Chebyshev norm; Bielecki norm.
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Chapter 1

Introduction and Preliminaries

The stability theory for functional equations started with a problem related to the stability of group homomorphism that was considered by Ulam in 1940 ([23]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces in [9]. Thereafter, this type of stability is called the Hyers-Ulam stability. In 1978, Th. M. Rassias [20] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. In fact, he has introduced a new type of stability which is called the Hyers-Ulam-Rassias stability.

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation [1]. Recently some authors ([10], [11], [21], [24] and [25]) extended
the Ulam stability problem from an integer-order differential equation to a fractional-order
differential equation.

Integral equations of various types play an important role in many branches of functional
analysis and in their applications, for example in physics, economics and other fields. Also,
the fractional differential equations are useful tools in the modelling of many physical phe-
nomena and processes in economics, chemistry, aerodynamics, etc. (for more details see
\[12, 14, 15, 22\]).

There are different types of fractional integral equations. In \[5\], authors by defining the
types of Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation proved that
every mapping from this type can be somehow approximated by an exact solution of the
considered equation.

Now we presented similar definitions with \[5\] and prove stability results for the following
deterministic semilinear fractional Volterra integral equation

\[
X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s))) \, ds, \quad t \geq 0
\]

that $\beta \in (0, 1)$, $A \leq 0$, and $h : \mathbb{R} \rightarrow \mathbb{R}$ and $\xi_t : \mathbb{R}_+ \rightarrow \mathbb{R}$ are two measurable functions.

Bao \[3\] used the Grinwall inequality to state the mean square stability for volterra-Ito
equations with a function as initial condition and bounded kernels. Several researchers have
studied stability of stochastic systems via Lyapunov function techniques \[13, 26, 27\]. Fiel et
al. in [6] investigated the asymptotic stability of the stochastic system driven by fractional Volterra integral equation. Here we consider the Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability of this stochastic systems driven by Brownian motion has also been studied.

S. M. ULAM was born in Lwow, Poland on April 3, 1909 and died in Santa Fe, U.S.A. on May 13, 1984. He graduated with a doctorate in pure mathematics from the Polytechnic Institute at Lwow in 1933. Ulam worked at: The Institute for Advanced Study, Princeton (1936), Harvard University (1939-40), University of Wisconsin (1941-43), Los Alamos Scientific Laboratory (1943-65), University of Colorado (1965-76), and University of Florida (1974). He was a member of the American Academy of Arts and Sciences and the National Academy of Sciences. He made fundamental contributions in mathematics, physics, biology, computer science, and the design of nuclear weapons. His early mathematical work was in set theory, topology, group theory, and measure theory. While still a schoolboy in Lwow, Ulam signed his notebook ”S. Ulam, astronomer, physicist and mathematician”. As Ulam notes, ”the aesthetic appeal of pure mathematics lies not merely in the rigorous logic of the proofs and theorems, but also in the poetic elegance and economy in articulating each step in a mathematical presentation.”

Functional Equations and Difference Inequalities and Ulam Stability Notions, is a forum
for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the first centennial birthday anniversary of Stanislaw Marcin ULAM.

This collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical and physical sciences.

Let $E$ and $F$ be Banach spaces. A mapping $f : E \to F$ is called additive function, if it satisfies the Cauchy functional equation $f(x + y) = f(x) + f(y)$ for all $x, y \in E$.

Ulam in 1940 in a talk given at Wisconsin University. The stability problem posed by Ulam was the following: Under what conditions there exists an additive mapping near an approximately additive mapping? (for more details see [23]). The first answer to the question of Ulam was given by Hyers [9] in 1941 in the case of Banach spaces: Let $X_1, X_2$ be two Banach spaces and $\varepsilon > 0$. Then for every mapping $f : X_1 \longrightarrow X_2$ satisfying $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X_1$, there exists a unique additive mapping $g : X_1 \longrightarrow X_2$ with the property $\|f(x) - g(x)\| \leq \varepsilon$, $\forall x \in X_1$.

This type of stability is called Hyers-Ulam stability. In 1978, Th. M. Rassias [20] provided a remarkable generalization of the Hyers-Ulam stability by considering variables on the right-hand side of the inequalities.

The modified Ulam’s stability problem with the generalization control function was proved by P. Gavruta [7] in the following way
Theorem 1. Let $E$ be a Banach space and let $\phi : E \times E \to [0, +\infty)$ be a function satisfying
\[
\psi(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \phi(2^k x, 2^k y) < \infty
\]
for all $x, y \in E$. If a function $f : E \to F$ satisfies the functional inequality
\[
||f(x + y) - f(x) - f(y)|| < \phi(x, y)
\]
for all $x, y \in E$. Then there exists a unique additive function $T : E \to F$ which satisfies
\[
||f(x) - T(y)|| \leq \psi(x, y)
\]
for all $x \in E$.

J. M. Rassias [16]-[19] solved the Ulam's problem for different mappings, in the following way:

Theorem 2. Let $X$ be a real normed linear space and let $Y$ be a real normed linear space.

Assume in addition that if $f : X \to Y$ is a mapping for which there exist constant $\delta > 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and $f$ satisfies the inequality
\[
||f(x + y) - f(x) - f(y)|| \leq \delta ||x||^p ||y||^q
\]
for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying
\[
||f(x) - L(x)|| \leq \frac{\delta}{||p-q||} ||x||^r
\]
for all \( x \in X \).

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the initial equation (see [16], [17], [18] and [19]). Now we replace the control function as following:

**Theorem 3.** Let \( A \) be a normed algebra, \( B \) a complete normed algebra and \( f : A \to B \) a mapping such that \( f(tx) \) is continuous in \( t \) for each fixed \( x \in A \). Assume that there exists \( \delta \geq 0 \) and \( p \neq 1 \) such that \( ||f(x+y) - f(x) - f(y)|| \leq \delta ||x+y||^p \) for every \( x, y \in A \). Then there exist unique linear mapping \( T : A \to B \) and \( \beta > 0 \) such that \( ||f(x) - T(x)|| \leq \beta ||x||^p \) for every \( x \in A \).

**Proof.** For \( p < 1 \) the function \( \varphi(x, y) = \delta ||x+y||^p \) is an admissible control function, so the proof is complete.

For \( p > 1 \), define the mapping \( T \) by the formula \( T(x) = \lim_{n \to \infty} 2^n f(x/2^n) \) for all \( x \in A \). Obviously, one has to verify the convergence of the sequence occurring on the right-hand side of \( T(x) = \lim_{n \to \infty} 2^n f(x/2^n) \). Putting \( x/2 \) in place of \( x \) and \( y \) in inequality \( ||f(x+y) - f(x) - f(y)|| \leq \delta ||x+y||^p \), we obtain \( ||f(x) - 2f(x/2)|| \leq \delta ||x||^p \) for all \( x \in A \). Hence for each \( n \in \mathbb{N} \) and every \( x \in A \), we have:
\[ ||f(x) - 2^n f(x/2^n)|| \leq ||f(x) - 2 f(x/2)|| + 2 ||f(x/2) - 2 f(x/2^2)|| + ... + 2^{n-1} ||f(x/2^{n-1}) - 2 f(x/2^n)|| \leq (1 + \frac{2}{2p - 2}) \delta ||x||^p = \beta ||x||^p \]

where \( \beta = (1 + \frac{2}{2p - 2}) \delta \). Now fix an \( x \in A \) and choose arbitrary \( m, n \in \mathbb{N} \) with \( m > n \). Then \( ||2^m f(x/2^m) - 2^n f(x/2^n)|| \leq 2^{n(1-p)} \beta ||x||^p \), which becomes arbitrary small as \( n \to \infty \). On account of the completeness of the algebra \( B \), this implies that the sequence \( \{2^n f(x/2^n) : n \in \mathbb{N} \} \) is convergent for each \( x \in A \). Thus \( T \) is correctly defined. Moreover, it satisfies in the condition \( ||f(x) - T(x)|| \leq \beta ||x||^p \) as \( n \to \infty \). It is sufficient to show that \( T \) is additive. Replacing \( x \) by \( x/2^n \) and \( y \) by \( y/2^n \) in \( ||f(x+y) - f(x) - f(y)|| \leq \delta ||x + y||^p \) and then multiplying both sides of the resulting inequality by \( 2^n \), we get

\[ ||2^n f(\frac{x+y}{2^n}) - 2^n f(x/2^n) - 2^n f(y/2^n)|| \leq 2^{n(1-p)} \delta ||x + y||^p \]

for all \( x, y \in A \). Since the right-hand side of this inequality tends to zero as \( n \to \infty \), it becomes apparent that the mapping \( T \) is additive. The function \( f(tx) \) is continuous relative to \( t \), so \( T(tx) \) is continuous in \( t \), and it is linear. For the uniqueness of \( T \), suppose that there exists another \( S \) such that \( ||f(x) - S(x)|| \leq \beta ||x||^p \) and \( S(x) \neq T(x) \). For any integer \( n > \frac{2 \beta ||x||^p}{||T(x) - S(x)||} \), it is obvious that \( ||T(nx) - S(nx)|| > 2 \beta ||x||^p \), which contradicts with the inequalities \( ||T(x) - f(x)|| \leq \beta ||x||^p \) and \( ||S(x) - f(x)|| \leq \beta ||x||^p \). Hence \( T \) is the unique linear map such that satisfies in \( ||f(x) - T(x)|| \leq \beta ||x||^p \). \( \square \)
The fixed point method for studying the Hyers-Ulam stability of functional equations was used for the first time in 1991. Namely, in [2], J.A. Baker proved the following variant of Banach’s fixed point theorem.

**Theorem 4.** Let \((Y,d)\) be a complete metric space and \(T : Y \to Y\) be a contraction (that is, there is a \(\lambda \in [0,1)\) such that

\[
d(T(x), T(y)) \leq \lambda d(x, y)
\]

for all \(x, y \in Y\). If \(u \in Y, \delta > 0\) and

\[
d(u, T(u)) \leq \delta
\]

then \(T\) has a unique fixed point \(p \in Y\). Moreover,

\[
d(u, p) \leq \frac{\delta}{1 - \lambda}
\]

Next, he applied Theorem 16 to obtain the following result concerning the stability of a quite general functional equation in a single variable.

**Theorem 5.** Let \(S\) be a nonempty set, \((X,d)\) be a complete metric space, \(\phi : S \to S, F : S \times X \to X, \lambda \in [0,1)\) and

\[
d(F(t, u), F(t, v)) \leq \lambda d(u, v) \quad \quad \quad t \in S, u, v \in X.
\]

If \(g : S \to X, \delta > 0\) and
then there is a unique function $f : S \to X$ such that

$$f(t) = F(t, f(\phi(t))) \quad t \in S$$

and

$$d(f(t), g(t)) \leq \frac{\delta}{1 - \lambda} \quad t \in S.$$

**Proof.** We put

$$Y = \{a : S \to X : \sup \{d(a(t), g(t)), t \in S\} < \infty\}$$

and

$$d'(a, b) = \sup \{d(a(t), b(t)), t \in S\}, \quad a, b \in Y.$$

Then $g \in Y$ and $(Y, d')$ is a complete metric space. Next, we define

$$T(a)(t) = F(t, a(\phi(t))), \quad a \in Y, t \in S$$

and show that

$$d'(T(a), T(b)) \leq \lambda d'(a, b), \quad a, b \in Y.$$  

Since $d'(g, T(g)) \leq \delta$, Theorem [16] finishes the proof. □

Theorem [5] with
\[ F(t, x) = \alpha(t) + \beta(t)x, \quad t \in S, x \in E \]

gives:

**Corollary 6.** Let \( S \) be a nonempty set, \( E \) be a real (or complex) Banach space, \( \phi : S \rightarrow S, \alpha : S \rightarrow E, \beta : S \rightarrow \mathbb{R}(or \mathbb{C}), \lambda \in [0, 1) \) and

\[ ||\beta(t)|| \leq \lambda, \quad t \in S. \]

If \( g : S \rightarrow E, \delta > 0 \) and

\[ ||g(t) - (\alpha(t) + \beta(t)g(\phi(t)))|| \leq \delta, \quad t \in S \]

then there exists a unique function \( f : S \rightarrow E \) such that

\[ f(t) = \alpha(t) + \beta(t)f(\phi(t)), \quad t \in S \]

and

\[ ||f(t) - g(t)|| \leq \frac{\delta}{1 - \lambda}, \quad t \in S. \]

Now, following \[ \text{[4]} \], we show how the stability of a functional equation can be deduced from the following variant of Banach’s fixed point theorem.

**Theorem 7.** If \( (Y, d) \) is a complete metric space and \( T : Y \rightarrow Y \) is a contraction (with a constant \( \lambda \)), then \( T \) has a unique fixed point \( p \in Y \). Moreover,

\[ d(u, p) \leq \frac{d(u, T(u))}{1 - \lambda}, \quad u \in Y. \]
**Theorem 8.** Let $S$ be a nonempty set, $(X, d)$ be a complete metric space, $\phi : S \to S$, $F : X \times X \to X$, $\lambda, \mu \in \mathbb{R}^+$ and

$$d(F(s, u), F(t, v)) \leq \mu d(s, t) + \lambda d(u, v), \quad s, t, u, v \in X.$$  

Assume also that $g : S \to X$, $\Phi : S \to \mathbb{R}^+$ are such that

$$d(g(t), F(g(t), g(\phi(t)))) \leq \Phi(t), \quad t \in S$$

and there exists an $L \in [0, 1)$ with

$$\lambda \Phi(\phi(t)) + \mu \phi(t) \leq L \Phi(t), \quad t \in S.$$  

Then there is a unique function $f : S \to X$ such that

$$f(t) = F(f(t), f(\phi(t))), \quad t \in S$$

and

$$d(f(t), g(t)) \leq \frac{\Phi(t)}{1 - L}, \quad t \in \mathbb{R}.$$  

**Proof.** To prove the theorem we put

$$Y = \{a : S \to X : \inf\{||k \in [0, \infty] : d(a(t), g(t)) \leq k \Phi(t), \quad t \in S\} < \infty\}$$

and

$$d(a, b) = \inf\{k \in [0, \infty] : d(a(t), b(t)) \leq k \Phi(t), \quad t \in S\}, \quad a, b \in Y.$$
Then \( g \in Y \) and \((Y, d)\) is a complete metric space. Moreover, the formula

\[
T(a)(t) = F(a(t), a(\phi(t))), \quad a \in Y, t \in S
\]

defines a mapping \( T : Y \to Y \) which is a contraction satisfying \( d(g, T(g)) \leq 1 \), and Theorem \( \square \) finishes the proof.

Let us mention here that Theorem \( \square \) was also applied in [8] to the proof of the following result.

**Theorem 9.** Let \( S \) be a nonempty set, \((X, d)\) be a complete metric space, \( \phi : S \to S, F : S \times X \to X, \alpha : S \to (0, \infty), \beta \in [0, 1) \), and for any \( t \in S, u, v \in X^S \),

\[
\alpha(\phi(t))d(F(t, u(\phi(t))), F(t, v(\phi(t)))) \leq \lambda \alpha(t)d(u(\phi(t)), v(\phi(t))).
\]

If \( g : S \to X \) satisfies the inequality

\[
d(g(t), F(t, g(\phi(t)))) \leq \alpha(t), \quad t \in S,
\]

then there is a unique function \( f : S \to X \) such that \( f(t) = F(t, f(\phi(t))) \) and

\[
d(f(t), g(t)) \leq \frac{\alpha(t)}{1-\lambda}, \quad t \in S.
\]

Now, we introduce notations, definitions and theorems which are used throughout this project.
Definition 10. The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ that $z, \alpha \in \mathbb{C}, \Re(\alpha) > 0$, defined as,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

where the Euler Gamma function $\Gamma : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} \exp(-s)ds.$$

Definition 11. The beta function is defined as

$$\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}dt$$

that $x, y \in \mathbb{C}$ and $\Re x > 0, \Re y > 0$.

From the definition of $\Gamma$ and $\beta$ functions we have

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

Definition 12. For a function $h$ given on the interval $[a, b]$, the $\alpha$th Riemann-Liouville fractional order derivative of $h$, is defined by

$$(D_a^\alpha h)(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t - s)^{(n-\alpha-1)} h(s)ds,$$  

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of $\alpha$.

Definition 13. For a function $h$ given on the interval $[a, b]$, the Caputo fractional order derivative of $h$, is defined by
\[ cD^n_{a^+} h(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} h^{(n)}(s)ds, \]

where \( n = [\alpha] + 1. \)

**Definition 14.** Given an interval \([a, b]\) of \(\mathbb{R}\), then the fractional order integral of a function

\[ h \in L^1([a, b], \mathbb{R}) \] of order \( \gamma \in \mathbb{R}_+ \) is defined by

\[ \Gamma_{a^+}^\gamma h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t - s)^{\gamma-1} h(s)ds, \]

where \( \Gamma(.) \) is the Gamma function.

In the sequel, we will use a Banach’s fixed point theorem in a framework of a generalized complete metric space. For a nonempty set \( X \), we introduce the definition of the generalized metric on \( X \).

**Definition 15.** A function \( d : X \times X \rightarrow [0, +\infty] \) is called a generalized metric on \( X \) if and only if satisfies

\[ (A_1) \quad d(x, y) = 0 \text{ if and only if } x = y; \]

\[ (A_2) \quad d(x, y) = d(y, x) \text{ for all } x, y \in X; \]

\[ (A_3) \quad d(x, z) \leq d(x, y) + d(y, z) \text{ for all } x, y, z \in X. \]

**Theorem 16.** Let \((X, d)\) be a generalized complete metric space. Assume that \( \Lambda : X \rightarrow X \) is a strictly contractive operator with the Lipschitz constant \( L < 1 \). If there exists a non-
negative integer \( k \) such that \( d(\Lambda^{k+1}x, \Lambda^k x) < \infty \) for some \( x \in X \), then the following are true:

(a) The sequence \( \Lambda^n x \) convergence to a fixed point \( x^* \) of \( \Lambda \);

(b) \( x^* \) is the unique fixed point of \( \Lambda \) in

\[
X^* = \{ y \in X | d(\Lambda^k x, y) < \infty \} \;
\]

(c) If \( y \in X^* \), then

\[
d(y, x^*) \leq \frac{1}{1 - L} d(\Lambda y, y). \]

In this section, we will study Mittag-Leffler-Hyers-Ulam stability of the following deterministic semilinear fractional Volterra integral equation

\[ X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds, \quad t \geq 0 \quad (1.0) \]

that \( \beta \in (0, 1), \ A < 0, \) and \( h: \mathbb{R} \to \mathbb{R} \) and \( \xi_t: \mathbb{R}^+ \to \mathbb{R} \) are two measurable functions.

**Definition 17.** Equation (1.0) is Mittag-Leffler-Hyers-Ulam stable if there exists a real number \( c > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( y \) of the inequality
there exists a unique solution \( y_0 \) of equation (1.0) satisfying the following inequality:

\[
|y(t) - y_0(t)| \leq \varepsilon E_\beta(t^\beta).
\]

**Theorem 18.** Suppose \( \beta \in (0, 1) \), \( A < 0 \), \( h : \mathbb{R} \to \mathbb{R} \) and \( \xi_t : \mathbb{R}^+ \to \mathbb{R} \) are two measurable functions, \( X(s) \leq s \), and

\[
|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s)))ds| \leq \varepsilon E_\beta(t^\beta)
\]

then equation (1.0) is Mittag-Leffler-Hyers-Ulam stable.

**Proof.** Let us consider the space of continuous functions

\[
X = \{ g : \mathbb{R} \to \mathbb{R} \mid g \text{ is continuous} \}.
\]

Similar to the Theorem 3.1 of [?] endowed with the generalized metric defined by

\[
d(g, h) = \inf\{ K \in [0, \infty) \mid |g(t) - h(t)| \leq K \varepsilon E_\beta(t^\beta) \text{ for all } t \in \mathbb{R} \}. \tag{2.0}
\]

for \( \varepsilon > 0 \). It is known that \((X, d)\) is a generalized complete metric space.

Define an operator \( \Lambda : X \to X \) by

\[
(\Lambda h)(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s)))ds, \tag{3.0}
\]
for all \( h \in X \) and \( t \in \mathbb{R} \).

Therefore the function \( h \) is continuous, so \( \Lambda h \) is also continuous and this ensures that \( \Lambda \) is a well defined operator. For any \( g, h \in X \), let \( K_{gh} \in [0, \infty] \) such that

\[
|g(t) - h(t)| \leq K_{gh} \varepsilon E_\beta(t^\beta) \tag{4.0}
\]

for any \( t \in \mathbb{R} \). This \( K_{gh} \) exists, because of definition of \((X, d)\). From the definition of \( \Lambda \) in (3.0) and (4.0) we have

\[
|(\Lambda g)(t) - (\Lambda h)(t)| =
\]

\[
|\xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + g(X(s))) ds - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s))) ds| =
\]

\[
= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [AX(s) + g(X(s)) - AX(s) - h(X(s))] ds
\]

\[
= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (g(X(s)) - h(X(s))) ds |
\]

\[
\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} K_{gh} \varepsilon E_\beta((X(s))^\beta) |ds |
\]

\[
= \frac{K_{gh} \varepsilon}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} E_\beta((X(s))^\beta) ds
\]

\[
= \frac{K_{gh} \varepsilon}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sum_{k=0}^\infty \frac{(X(s))^{k\beta}}{\Gamma(k\beta + 1)} ds
\]

\[
= \frac{K_{gh} \varepsilon}{\Gamma(\beta)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\beta + 1)} \int_0^t (t-s)^{\beta-1} (X(s))^{k\beta} ds
\]
\[ \leq K_{gh} \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\beta + 1)} \int_{0}^{t} (t-s)^{\beta - 1}(s)^{k\beta} \, ds \]

\[ = K_{gh} \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\beta + 1)} \int_{0}^{t} (t-tx)^{\beta - 1}(tx)^{k\beta} \, dx \]

\[ = K_{gh} \varepsilon \sum_{k=0}^{\infty} \frac{t^{(k+1)\beta}}{\Gamma(k\beta + 1)} \int_{0}^{t} (1-x)^{\beta - 1}(t)^{k\beta} \, dx \]

\[ = K_{gh} \varepsilon \sum_{k=0}^{\infty} \frac{t^{(k+1)\beta}}{\Gamma((k+1)\beta + 1)} \]

\[ \leq K_{gh} \varepsilon \sum_{n=0}^{\infty} \frac{t^{n\beta}}{\Gamma(n\beta + 1)} = K_{gh} \varepsilon E_{\beta}(t^{\beta}), \]

for all \( t \in \mathbb{R} \); that is \( d(\Lambda g, \Lambda h) \leq K_{gh} \varepsilon E_{\beta}(t^{\beta}) \). Hence, we can conclude that \( d(\Lambda g, \Lambda h) \leq d(g, h) \) for any \( g, h \in X \), and so the strictly continuous property is verified.

Let us take \( g_{0} \in X \). From the continuous property of \( g_{0} \) and \( \Lambda g_{0} \), it follows that there exists a constant \( 0 < K_{1} < \infty \) such that

\[ |(\Lambda g_{0})(t) - g_{0}(t)| = |\xi_{t} - \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta - 1}(AX(s) + g_{0}(X(s))) \, ds - g_{0}(t)| \leq K_{1} \varepsilon E_{\beta}(t^{\beta}), \]

for all \( t \in \mathbb{R} \), since \( g_{0} \) is bounded on \( \mathbb{R} \) and \( \min_{t \in \mathbb{R}} E_{\beta}(t^{\beta}) > 0 \), thus, (2.0) implies that \( d(\Lambda g_{0}, g_{0}) < \infty \). Therefore, according to Theorem 16, there exists a continuous function \( y_{0} : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \Lambda^{n} g_{0} \rightarrow y_{0} \) in \( (X, d) \) as \( n \rightarrow \infty \) and \( \Lambda y_{0} = y_{0} \); that is, \( y_{0} \) satisfies the equation (1.0) for every \( t \in \mathbb{R} \).
We will now prove that \( \{g \in X | d(g_0, g) < \infty\} = X \) for any \( g \in X \), since \( g \) and \( g_0 \) are bounded in \( \mathbb{R} \) and \( \min_{t \in \mathbb{R}} E_{\beta}(t^\beta) > 0 \), there exists a constant \( 0 < C_g < \infty \) such that

\[
|g_0(t) - g(t)| \leq C_g E_{\beta}(t^\beta),
\]

for any \( t \in \mathbb{R} \). Hence, we have \( d(g_0, g) < \infty \) for all \( g \in X \); that is,

\[
\{g \in X | d(g_0, g) < \infty\} = X.
\]

Hence, in view of theorem 16, we conclude that \( y_0 \) is the unique continuous function which satisfies the equation (1.0). Now we have \( d(y, \Lambda y) \leq \varepsilon E_{\beta}(t^\beta) \). Finally, Theorem 16 together with the above inequality imply that

\[
d(y, y_0) \leq \frac{1}{1 - L} d(y, \Lambda y) \leq \frac{1}{1 - L} \varepsilon E_{\beta}(t^\beta).
\]

This means that the equation (1.0) is Mittag-Leffler-Hyers-Ulam stable.

Next, we use the Chebyshev norm \( \| \cdot \|_c \) to derive the above similar result for the equation (1.0).

**Theorem 19.** Suppose \( \beta \in (0, 1) \), \( A < 0 \), \( h : \mathbb{R} \rightarrow \mathbb{R} \) and \( \xi_t : \mathbb{R}^+ \rightarrow \mathbb{R} \) are two measurable functions, with \( X(s) \leq s \).

Then the equation (1.0) is Mittag-Leffler-Hyers-Ulam stable via the Chebyshev norm.
Proof. Just like the discussion in Theorem 3.1, we prove that $\Lambda$ defined in (3.0) is a contraction map on $X$ with respect to the Chebyshev norm:

$$|(\Lambda g)(t) - (\Lambda h)(t)| =$$

$$= \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} (g(X(s)) - h(X(s))) ds$$

$$\leq \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} |g(X(s)) - h(X(s))| ds$$

$$\leq \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \left( \max_{s \in [0,t]} |g(X(s)) - h(X(s))| \right) ds$$

$$\leq \frac{\|g - h\|_c}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} ds$$

$$= \frac{\|g - h\|_c}{\Gamma(\beta)} \frac{t^\beta}{\beta} = \frac{t^\beta}{\Gamma(\beta + 1)} \|g - h\|_c \leq E_\beta(t),$$

for all $t \in \mathbb{R}$; that is $d(\Lambda g, \Lambda h) \leq \|g - h\|_c E_\beta(t)$. Hence, we can conclude that $d(\Lambda g, \Lambda h) \leq E_\beta(t)d(g, h)$ for any $g, h \in X$. By letting $0 < E_\beta(t) < 1$ the strictly continuous property is verified. Now by a similar process with Theorem 3.1, we have

$$d(y, y_0) \leq \frac{1}{1 - E_\beta(t)} d(\Lambda y, y) \leq \frac{1}{1 - E_\beta(t)} \varepsilon E_\beta(t^\beta) \leq C\varepsilon E_\beta(t^\beta)$$

which means that equation (1.0) is Mittag-Leffler-Hyers-Ulam stable via the Chebyshev norm.

In the following Theorem we have used the Bielecki norm

$$\|g\|_B := \max_{t \in [a, b]} |g(t)|e^{-\theta t}, \quad \theta > 0, \quad a, b \in \mathbb{R}$$

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to derive the similar Theorem 3.1 for the fundamental equation (1.0) via the Bielecki norm.

**Theorem 20.** suppose that $\beta \in (0, 1)$, $A < 0$, $h : \mathbb{R} \rightarrow \mathbb{R}$ and $\xi_t : \mathbb{R}^+ \rightarrow \mathbb{R}$ are two measurable functions with $X(s) \leq s$.

Then the equation (1.0) is Mittag-Leffler-Hyers-Ulam stable via the Bielecki norm.

**Proof.** Just like the discussion in Theorem 3.1, we prove that $\Lambda$ defined in (3.0) is a contraction on $X$ with respect to the Bielecki norm:

$$
|\Lambda g(t) - \Lambda h(t)| =
$$

$$
= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (g(X(s)) - h(X(s))) ds
$$

$$
\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |g(X(s)) - h(X(s))| ds
$$

$$
\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} e^{g_s} \left( \max_{s \in [0,t]} |g(X(s)) - h(X(s))| e^{-g_s} \right) ds
$$

$$
\leq \frac{\|g - h\|_B}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} e^{g_s} ds
$$

$$
\leq \frac{\|g - h\|_B}{\Gamma(\beta)} \left( \int_0^t (t-s)^{2(\beta-1)} ds \right)^{\frac{1}{2}} \left( \int_0^t e^{2g_s} ds \right)^{\frac{1}{2}}
$$

$$
= \frac{\|g - h\|_B}{\Gamma(\beta)} \cdot \frac{t^\beta}{\sqrt{2\beta - 1}} \cdot \frac{e^{\theta t}}{\sqrt{2}},
$$

for all $t \in \mathbb{R}$; that is $d(\Lambda g, \Lambda h) \leq \frac{t^\beta \cdot e^{\theta t}}{\Gamma(\beta) \cdot \sqrt{2(2\beta - 1)}} \| g - h \|_B$. Hence, we can conclude that

$$
d(\Lambda g, \Lambda h) \leq \frac{t^\beta \cdot e^{\theta t}}{\Gamma(\beta) \cdot \sqrt{2(2\beta - 1)}} \| g - h \|_B.
$$
for any \( g, h \in X \), by letting \( 0 < \frac{t^\beta e^{\theta t}}{\Gamma(\beta) \sqrt{2(2\beta - 1)\theta}} < 1 \) the strictly continuous property is verified. Now by a similar process with Theorem 3.1, we have

\[
d(y, y_0) \leq \frac{1}{1 - \frac{t^\beta e^{\theta t}}{\Gamma(\beta) \sqrt{2(2\beta - 1)\theta}}} d(Ay, y) \leq \frac{1}{1 - \frac{t^\beta e^{\theta t}}{\Gamma(\beta) \sqrt{2(2\beta - 1)\theta}}} \varepsilon E_\beta(t^\beta) \leq C \varepsilon E_\beta(t^\beta)
\]

which means that equation (1.0) is Mittag-Leffler-Hyers-Ulam stable via the Bielecki norm.
Chapter ٣

Mittag-Leffler-Hyers-Ulam-Rassias and asymptotic stability

At first we introduce the concept of Mittag-Leffler-Hyers-Ulam-Rassias stability and then we prove the equation (1.0) has the Mittag-Leffler-Hyers-Ulam-Rassias stability.

**Definition 21.** Equation (1.0) is Mittag-Leffler-Hyers-Ulam-Rassias stable if there exists a real number \( c > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( y \) of the inequality

\[
|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds| \leq \varphi(t)\varepsilon E_\beta(t^\beta)
\]
there exists a unique solution $y_0$ of equation $1.0$ satisfying the following inequality:

$$|y(t) - y_0(t)| \leq c\varphi(t)\varepsilon E_\beta(t^\beta).$$

where $\varphi : X \to [0, \infty)$ is a continuous function.

**Theorem 22.** Put $M = \frac{1}{\Gamma(\beta)} \left( \frac{1-p}{\beta-p} \right)^{1-p} t^{\beta-p}$ with $0 < p < \beta$. Let $0 < K M < 1$. Suppose that $\beta \in (0, 1)$, $A < 0$, $h : \mathbb{R} \to \mathbb{R}$ and $\xi_t : \mathbb{R}^+ \to \mathbb{R}$ are two measurable functions, $\varphi(X(s)) \leq \varphi(s)$ and

$$|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s)) ds| \leq \varepsilon \varphi(t) E_\beta(t^\beta)$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a $L_\frac{1}{p}$-integrable function which satisfies

$$\left( \int_0^t (\varphi(s))^{\frac{1}{p}} ds \right)^p \leq K \varphi(t)$$

for all $t \in \mathbb{R}$. Then equation $1.0$ is Mittag-Leffler-Hyers-Ulam-Rassias stable.

**Proof.** Let us consider the space of continuous functions

$$X = \{ g : [0, a] \to \mathbb{R} | g \text{ is continuous} \}.$$

endowed with the generalized metric defined by

$$d(g, h) = \inf\{ K \in [0, \infty] | |g(t) - h(t)| \leq K \varepsilon \varphi(t) \text{ for all } x \in \mathbb{R} \}. \quad (1.0)$$

for $\varepsilon > 0$. It is known that $(X, d)$ is a generalized complete metric space.
Define an operator $\Lambda : X \rightarrow X$ by

$$
(\Lambda h)(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}(AX(s) + h(X(s))) ds,
$$

(2.0)

for all $h \in X$ and $x \in \mathbb{R}$.

The function $h$ is continuous, so $\Lambda h$ is also continuous and this ensures that $\Lambda$ is a well defined operator. For any $g, h \in X$, let $K_{gh} \in [0, \infty]$ such that

$$
|g(t) - h(t)| \leq K_{gh} \mathcal{E}(\varphi(t))
$$

(3.0)

for any $t \in \mathbb{R}$. This $K_{gh}$ exists, because of definition of $(X, d)$. From the definition of $\Lambda$ in (2.0) and (3.0) we have

$$
|((\Lambda g)(t) - (\Lambda h)(t)| =

|\xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}(AX(s) + g(X(s))) ds - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}(AX(s) + h(X(s))) ds |

= \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}[AX(s) + g(X(s)) - AX(s) - h(X(s))] ds |

= \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}[g(X(s)) - h(X(s))] ds |

\leq \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} K_{gh} \mathcal{E}(\varphi(x(s))) ds |

= \frac{K_{gh} \mathcal{E}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \varphi(X(s)) ds |

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\[
\leq \frac{K_{gh}}{\Gamma(\beta)} \left| \int_0^t (t - s)^{\beta - 1} \varphi(s) ds \right|
\]
\[
\leq \frac{K_{gh}}{\Gamma(\beta)} \left( \int_0^t (t - s)^{\frac{\alpha}{p} - 1} ds \right)^{1 - p} \left( \int_0^t \varphi(s)^{\frac{1}{p}} ds \right)^p
\]
\[
\leq \frac{K_{gh}}{\Gamma(\beta)} \left( \frac{1 - p}{\beta - p} \right)^{1 - p} t^{\beta - p} K_0(t) = K_{gh} MK_0(t).
\]

for all \( t \in \mathbb{R} \); that is \( d(\Lambda g, \Lambda h) \leq K_{gh} MK_0(t) \). Hence, we can conclude that \( d(\Lambda g, \Lambda h) \leq MK_0 d(g, h) \) for any \( g, h \in X \), and so the strictly continuous property is verified.

Let us take \( g_0 \in X \). From the continuity property of \( g_0 \) and \( \Lambda g_0 \), it follows that there exists a constant \( 0 < K_1 < \infty \) such that

\[
|(\Lambda g_0)(x) - g_0(x)| = |\xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} (AX(s) + g(X(s))) ds - g_0(x)| \leq K_1 \varphi(t),
\]

for all \( t \in \mathbb{R} \), since \( g \) is bounded on \( \mathbb{R} \) and \( \min_{t \in \mathbb{R}} \varphi(t) > 0 \). Thus, (1.0) implies that \( d(\Lambda g_0, g_0) < \infty \). Therefore, according to Theorem 2.1, there exists a continuous function \( y_0 : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \Lambda^n g_0 \rightarrow y_0 \) in \((X, d)\) as \( n \rightarrow \infty \) and \( \Lambda y_0 = y_0 \); that is, \( y_0 \) satisfies the equation (1) for every \( t \in \mathbb{R} \).

We will now prove that \( \{ g \in X | d(g_0, g) < \infty \} = X \). for any \( g \in X \), since \( g \) and \( g_0 \) are bounded in \( \mathbb{R} \) and \( \min_{t \in \mathbb{R}} \varphi(t) > 0 \), there exists a constant \( 0 < C_g < \infty \) such that

\[
|g_0(t) - g(t)| \leq C_g \varphi(t),
\]

for any \( t \in \mathbb{R} \). Hence, we have \( d(g_0, g) < \infty \) for all \( g \in X \); that is,

\[
\{ g \in X | d(g_0, g) < \infty \} = X.
\]

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Hence, in view of theorem 2.5, we conclude that \( y_0 \) is the unique continuous function which satisfies the equation (1.0). Now we have \( d(y, \Lambda y) \leq \varepsilon MK \). Finally, Theorem 2.5 together with the above inequality imply that

\[
d(y, y_0) \leq \frac{1}{1 - LMK} d(\Lambda y, y) \leq \frac{1}{1 - L}.
\]

This means that the equation (1.0) is Mittag-Leffler-Hyers-Ulam stable.

Here we introduce the Young integral, which is an integral with respect to Holder continuous functions.

**Definition 23.** For \( T > 0 \) and \( \gamma \in (0, 1) \), let \( C_1^\gamma([0, T]; \mathbb{R}) \) be the set of \( \gamma \)-Holder continuous functions \( g : [0, T] \to \mathbb{R} \) of one variable such that the seminorm

\[
\| g \|_{\gamma, [0, T]} := \sup_{r \neq t, t \in [0, t]} \left| \frac{g_t - g_r}{|t - r|^{\gamma}} \right|^r,
\]

is finite. Also by \( \| g \|_{\infty, [0, T]} \) we denote the supremum norm of \( g \).

**Definition 24.** The noise is an additive solution for linear stochastic differential equations and has form \( \int_0^T \sigma(s) dW_s^H \). Here

\[
W_t^H = \int_0^t (t - s)^{H-1/2} dW_s, \quad H \in (1/2, 1)
\]

\( W \) is a Brownian motion and \( \sigma \) is a deterministic function such that

\[
\int_0^\infty \sigma^2(s) e^{2\lambda s} ds < \infty
\]
for some $\lambda > 0$.

Now we have following results about the semilinear fractional Volterra integral equation with additive noise:

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AX(s) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\theta_s, \quad t \geq 0 \quad (4.0)$$

where the initial condition $\xi = \{\xi_t; t \geq 0\}$ is bounded on compact sets and measurable, $\beta \in (0, 1), A \in \mathbb{R}, \alpha \in (1, 2), \theta = \{\theta_s, s \geq 0\}$ is a $\gamma$- Holder continuous function with $\gamma \in (0, 1)$ and $\Gamma$ is the Gamma function.

The second integral in (4.0) is a Young one and it is well-defined if $\alpha-1+\gamma > 1$, because $s \mapsto (t-s)^{\alpha-1}$ is $(\alpha - 1)$- Holder continuous on $[0; T]$.

**Corollary 25.** suppose $\xi = \{\xi_t; t \geq 0\}$ is bounded on compact sets and measurable, $\beta \in (0, 1), A \in \mathbb{R}, \alpha \in (1, 2), h : \mathbb{R} \to \mathbb{R}$ is measurable function, $\theta = \{\theta_s, s \geq 0\}$ is a $\gamma$- Holder continuous function with $\gamma \in (0, 1)$ and $\Gamma$ is the Gamma function, and

$$|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s))) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\theta_s| \leq \varepsilon \varepsilon_{\beta}(t)$$

then equation (4.0) is Mittag-Leffler-Hyers-Ulam stable.

**Corollary 26.** suppose $\xi = \{\xi_t; t \geq 0\}$ is bounded on compact sets and measurable, $\beta \in (0, 1), A \in \mathbb{R}, \alpha \in (1, 2), h : \mathbb{R} \to \mathbb{R}$ is measurable function, $\theta = \{\theta_s, s \geq 0\}$ is a $\gamma$-
Holder continuous function with \( \gamma \in (0, 1) \) and \( \Gamma \) is the Gamma function, and

\[
|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s))) \, ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, d\theta_s| \leq \varepsilon \varphi(t) E_\beta(t^\beta)
\]

then equation (4.0) is Mittag-Leffler-Hyers-Ulam-Rassias stable.

The concept of the asymptotic stability of a solution \( y(t) \) of equation (1.0) was considered in [6]. Now we consider this concept by following definition:

**Definition 27.** For any \( \varepsilon > 0 \) there exist \( T(\varepsilon) > 0 \) and \( r(\varepsilon) > 0 \) such that, if \( x, y \in B_r \) and \( x(t), y(t) \) are solutions of equation (1), then \( |x(t) - y(t)| \leq \varepsilon \) for \( t \geq T(\varepsilon) \).

**Theorem 28.** Suppose \( \beta \in (0, 1) \), \( A < 0 \), \( h : \mathbb{R} \rightarrow \mathbb{R} \) and \( \xi_t : \mathbb{R}^+ \rightarrow \mathbb{R} \) are two measurable functions, and there exists \( M > 0 \) such that \( |h(x)| \leq M|x| \).

Then the equation (1.0) is asymptotically stable.

**Proof.** Put \( r = \frac{\beta \Gamma(\beta)}{2(|A| + M)t^\beta} \varepsilon \), suppose \( X(t) \in B_r \) and \( Y(t) \in B_r \), so \( |X(t) - Y(t)| < 2r \),

\[
|(X)(t) - (Y)(t)| = \\
\left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(X(s))) \, ds - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AY(s) + h(Y(s))) \, ds \right| \\
\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |AX(s) - AY(s)| + |h(X(s) - Y(s))| \, ds \\
\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left( |A| |X(s) - Y(s)| + M |X(s) - Y(s)| \right) \, ds
\]
\[
\leq \frac{|A| \cdot 2r + M \cdot 2r}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} ds \\
= \frac{2r(|A| + M)}{\Gamma(\beta)} \cdot \frac{t^\beta}{\beta} = \varepsilon
\]
References


Abstract.

In this project, we define and investigate Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassiass stability of deterministic semilinear fractional Volterra integral equation. Also, we prove that this equation is stable with respect to the Chebyshev and Bielecki norms. The stability of stochastic systems driven by Brownian motion has also been studied.

Keywords: Mittag-Leffler-Hyers-Ulam stability; Mittag-Leffler-Hyers-Ulam-Rassiass stability; deterministic Volterra integral equation; Chebyshev norm; Bielecki norm.
از داشته‌بودن صحیح ادبی با خاطرات جایت از این طرح نوشته قدردانی می‌کنم.
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